

Global solution to the drift-diffusion-Poisson system for semiconductors with nonlinear recombination-generation rate

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Abstract

In this paper, we study the Cauchy problem of a time-dependent drift-diffusion-Poisson system for semiconductors. Existence and uniqueness of global weak solutions are proven for the system with a higher-order nonlinear recombination-generation rate R . We also show that the global weak solution will converge to a unique equilibrium as time tends to infinity.

Keywords: drift-diffusion-Poisson system; global existence and uniqueness; long-time behavior.

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1 Introduction

We consider the following drift-diffusion-Poisson model for semiconductors that is a coupled system of parabolic-elliptic equations:

$$\begin{cases} n_t = \operatorname{div}(\nabla n + n\nabla(\psi + V_n)) - R(n, p, x), \\ p_t = \operatorname{div}(\nabla p + p\nabla(-\psi + V_p)) - R(n, p, x), \\ -\varepsilon^2 \Delta \psi = n - p - D(x). \end{cases} \quad (1.1)$$

System (1.1) models the transport of the electrons and holes in semiconductor and plasma devices (cf. [17, 18]). $n = n(x, t)$ is the spatial distribution of electrons (negatively charged) and $p = p(x, t)$ is the spatial distribution of holes (positively charged). $\psi = \psi(x, t)$ is the self-consistent electrostatic potential created by the two charge carrier species (electrons and holes) and by the doping profile $D = D(x)$ of the semiconductor device. The charge carriers are assumed to be confined by the external potentials V_n and V_p . This replaces

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the usual assumption of a bounded domain (cf. [8, 13, 18] and the references therein). The function $R = R(n, p, x)$ represents the so-called recombination-generation rate for electron and hole. The parameter ε appearing in the Poisson equation is the scaled Debye length of the semiconductor device that stands for the screening of the hole and electron particles. In this paper, we are interested in the Cauchy problem to system (1.1) and assume that (1.1) is subject to the following initial data

$$n(x, t)|_{t=0} = n_I(x) \geq 0, \quad p(x, t)|_{t=0} = p_I(x) \geq 0. \quad (1.2)$$

The generation and recombination of electrons and holes in a semiconductor play an important role in their electrical and optical behavior [12]. Recombination is a process by which both carriers annihilate each other: the electrons fall in one or multiple steps into the empty state that is associated with the hole. Generation can be viewed as its inverse process whereby electrons and holes are created. There are several typical recombination mechanisms that the energy of carriers will be dissipated during these processes by different ways (cf. e.g., [12, 17, 18]). For instance,

1. *Band-to-band recombination* (also referred to as direct thermal recombination). The energy is emitted in the form of a photon. The recombination rate depends on the density of available electrons and holes and it can be expressed as

$$R(n, p) = C(np - n_i^2), \quad (1.3)$$

where n_i denotes the intrinsic carrier density of the semiconductor.

2. *Shockley-Read-Hall (SRH) recombination* (also called the trap-assisted recombination). A two-step transition of an electron from the conduction band to the valence band occurs and R is in the form

$$R(n, p) = \frac{(np - n_i^2)}{r_1 n + r_2 p + r_3}, \quad (1.4)$$

where r_1, r_2, r_3 are proper positive functions.

3. *Auger recombination*. An electron and a hole recombine in a band-to-band transition, but the resulting energy is given off to another electron or hole in the form of kinetic energy. The corresponding recombination rate is similar to that of band-to-band recombination, but involves a third particle:

$$R(n, p) = (C_n n + C_p p)(np - n_i^2). \quad (1.5)$$

Extensive mathematical study of the drift-diffusion-Poisson system has been developed in the literature. For the initial boundary value problem of (1.1) in a bounded domain $\Omega \subset \mathbb{R}^N$ with various boundary conditions (e.g., the Neumann type or no-flux boundary conditions), existence and uniqueness as well as long-time behavior have been investigated by many authors, see for instance, [3–5, 7–9, 13–15, 26] and reference therein.

For the sake of modeling simplicity and for the particularly interesting mathematical features, it is also interesting to consider the Cauchy problem of (1.1). Existence and uniqueness results and stability of strong solutions in $L^p(\mathbb{R}^N)$ spaces ($N \geq 2$) were proven in [20] for a system analogous to (1.1). However, in their system there were no external potentials and the recombination-generation rate R was replaced by a given function $f = f(t, x)$ that expressed the variation of the charge by the external current. As far as the long-time behavior of global solutions to the Cauchy problem is concerned, when the recombination-generation term R is absent, exponential convergence to equilibrium with a confining potential and an algebraic rate towards a self-similar state without confinement have been obtained in [1] (cf. also [4]). The analysis therein is based on the well-known entropy approach for diffusion and diffusion-convection equations that has been extensively studied in recent years (cf. [2, 6] and the references therein). We also refer to [19] in which an optimal L^p decay estimate of solutions was obtained via a time weighted L^p energy method (without confinement and recombination-generation rate R). When the recombination-generation process is taken into account, the situation is more complicated. In [24], the authors proved the global existence and uniqueness of weak solutions of problem (1.1) in \mathbb{R}^3 with an (unbounded) external potential $V_n = V_p = V$ and under the restrictive assumption that R has a linear growth (which, however, recovers the Shockley-Read-Hall recombination, cf. (1.4)). Besides, existence and uniqueness of the steady state and partial result on the convergence to equilibrium were obtained. Recently, exponential L^1 convergence to equilibrium was proved in [10] via entropy method for global solutions to a simplified convection-diffusion-reaction model with confinement and Shockley-Read-Hall recombination-generation rate but neglecting the influence of the self-consistent potential ψ . It would be interesting to study the well-posedness as well as long time behavior of the full convection-diffusion-reaction-Poisson system (1.1)–(1.2) with more general recombination-generation rate R including the higher nonlinear cases (1.3) and (1.5).

For the sake of simplicity we consider the whole-space case posed on \mathbb{R}^3 . Similar results can be obtained for the two-dimensional whole space case with some minor modifications, due to the different properties of the Newtonian potential. Next, we make the following assumptions on confining potentials V_n, V_p , the recombination-generation rate R and the doping profile D :

(H1a) There exist constants $\rho_n, \rho_p > 0$ such that

$$\frac{\partial^2 V_n}{\partial x^2} \geq \rho_n \mathbb{I}, \quad \frac{\partial^2 V_p}{\partial x^2} \geq \rho_p \mathbb{I}, \quad \forall x \in \mathbb{R}^3,$$

in the sense of positive-defined matrix. Moreover, there exists $K > 0$ such that

$$\|\Delta V_n\|_{L^\infty(\mathbb{R}^3)} \leq K, \quad \|\Delta V_p\|_{L^\infty(\mathbb{R}^3)} \leq K. \quad (1.6)$$

(H1b) There exists $K' > 0$ such that

$$\|V_n(x) - V_p(x)\|_{L^\infty(\mathbb{R}^3)} \leq K' < +\infty. \quad (1.7)$$

(H2a) The recombination-generation rate $R = R(n, p, x)$ is of the form

$$R(n, p, x) = F(n, p) (np - \delta^2 \mu_n \mu_p),$$

where $\mu_n = e^{-V_n(x)}$, $\mu_p = e^{-V_p(x)}$. δ is a positive constant (the scaled average intrinsic carrier density of the semiconductor), which without loss of generality, can be rescaled as $\delta = 1$.

(H2b) The scalar function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a Lipschitz continuous function with linear growth, namely, there exist constants $c_1, c_2 > 0$ independent of n, p such that

$$\begin{aligned} |F(n_1, p_1) - F(n_2, p_2)| &\leq c_1(|n_1 - n_2| + |p_1 - p_2|), \quad \forall n_1, p_1, n_2, p_2 \in \mathbb{R}, \\ |F(n, p)| &\leq c_2(1 + |n| + |p|) \quad \forall n, p \in \mathbb{R}. \end{aligned} \quad (1.8)$$

Moreover, $F(n, p) \geq 0$ if $n, p \geq 0$.

(H3) $D(x) \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$.

Remark 1.1. It easily follows from (H1a) that $V_n(x)$ and $V_p(x)$ are uniformly convex and can be bounded from below by a finite number $V_b \in \mathbb{R}$. Thus, $\|\mu_n\|_{L^\infty} \leq e^{V_b}$ and $\|\mu_p\|_{L^\infty} \leq e^{V_b}$. Without loss of generality, we normalize the confining potentials that

$$\int_{\mathbb{R}^3} \mu_n dx = \int_{\mathbb{R}^3} \mu_p dx = 1.$$

Besides, we infer from (1.7) that the norms on $L^2(\mathbb{R}^3, e^{V_n(x)} dx)$ and $L^2(\mathbb{R}^3, e^{V_p(x)} dx)$ are equivalent.

Remark 1.2. In the following analysis, we set $\varepsilon = 1$ without loss of generality. We note that the quasi-neutral limit (namely, zero-Debye-length limit $\varepsilon \rightarrow 0$) of drift-diffusion-Poisson system is a challenging and physically complex modeling problem for bipolar kinetic models of semiconductors, which has been analyzed by many authors, see, e.g. [16, 23] and the references cited therein.

Now we state the the main results of this paper.

Theorem 1.1 (Well-posedness). *Suppose that (H1a)–(H3) are satisfied. For any initial data $n_I \in L^2(\mathbb{R}^3, e^{V_n(x)} dx) \cap L^\infty(\mathbb{R}^3)$, $p_I \in L^2(\mathbb{R}^3, e^{V_p(x)} dx) \cap L^\infty(\mathbb{R}^3)$, $n_I, p_I \geq 0$, problem (1.1)–(1.2) admits a unique global weak solution (n, p, ψ) such that for any $T > 0$,*

$$\begin{aligned} n &\in L^\infty(0, T; L^2(\mathbb{R}^3, e^{V_n(x)} dx) \cap L^\infty(\mathbb{R}^3)), \quad \nabla n \in L^2(0, T; \mathbf{L}^2(\mathbb{R}^3, e^{V_n(x)} dx)), \\ p &\in L^\infty(0, T; L^2(\mathbb{R}^3, e^{V_p(x)} dx) \cap L^\infty(\mathbb{R}^3)), \quad \nabla p \in L^2(0, T; \mathbf{L}^2(\mathbb{R}^3, e^{V_p(x)} dx)), \\ n_t, p_t &\in L^2(0, T; (H^1(\mathbb{R}^3))'), \\ n(t) &\geq 0, \quad p(t) \geq 0, \quad t \in [0, T], \quad a.e. \ x \in \mathbb{R}^3, \\ \nabla \psi &\in L^\infty(0, T; \mathbf{L}^\infty(\mathbb{R}^3)), \quad \Delta \psi \in L^\infty(0, T; L^\infty(\mathbb{R}^3)), \end{aligned}$$

where $\psi = \psi(t, x)$ is given by the Newtonian potential

$$\psi(t, x) = \frac{1}{S_3} \int_{\mathbb{R}^3} \frac{n(t, y) - p(t, y) - D(y)}{|x - y|} dy,$$

with $S_3 = \frac{2\pi^{\frac{3}{2}}}{\Gamma(\frac{3}{2})}$ being the surface area of the 2D unit ball.

Theorem 1.2 (Long-time behavior). *Under the assumptions in Theorem 1.1, we have the following uniform-in-time estimate for the global solutions:*

$$\|n(t)\|_{L^\infty(0, +\infty; L^r(\mathbb{R}^3))} + \|p(t)\|_{L^\infty(0, +\infty; L^r(\mathbb{R}^3))} < \infty, \quad \forall r \in [1, +\infty]. \quad (1.9)$$

Moreover, for every fixed $t^* > 0$, the global shifted solution $(n(t + s), p(t + s), \psi(t + s))$ ($s \in (0, t^*)$) of problem (1.1)–(1.2) converges to the unique steady state $(n_\infty, p_\infty, \psi_\infty)$ that satisfies (4.3) as $t \rightarrow +\infty$ in the following sense:

$$\begin{aligned} n(t + \cdot) &\rightarrow n_\infty, \quad p(t + \cdot) \rightarrow p_\infty \quad \text{in } L^1((0, t^*) \times \mathbb{R}^3), \\ \nabla \psi(t + \cdot) &\rightharpoonup \nabla \psi_\infty \quad \text{in } L^2(0, t^*; \mathbf{H}^1(\mathbb{R}^3)), \\ \psi(t + \cdot) &\rightharpoonup \psi_\infty \quad \text{in } L^2(0, t^*; L^6(\mathbb{R}^3)). \end{aligned}$$

As we have mentioned before, for a class of recombination-generation rate with at most linear growth, existence of global weak solutions to problem (1.1)–(1.2) has been obtained in [24]. However, argument therein fails to apply in our present case due to the higher-order nonlinear reaction term R that includes both the band-to-band and the Auger recombination (cf. (H2a)–(H2b)). On the other hand, well-posedness results for drift-diffusion-Poisson system with higher-order recombination-generation rate have been proved in the bounded domain case (see [7–9] for the case with F being bounded, and [26] for the case that F has a linear growth). Since now we are considering the Cauchy problem in the whole space that is unbounded and the carriers are confined by unbounded external potentials, the methods for the initial boundary value problem cannot be used directly. We need to exploit and develop several techniques in the literature to prove the global existence and uniqueness of solutions to problem (1.1)–(1.2). In order to overcome the difficulties from the higher-order reaction term R , we first introduce a L^∞ cut-off to the unknowns n, p in R and study an approximation problem associated with our original system (1.1)–(1.2). To deal with the unbounded confining potentials, we then transform the approximate problem into a new form by introducing some new variables with proper weight functions. After obtaining the well-posedness of the approximate problem, we try to derive proper uniform estimates based on a Stampacchia-type L^∞ estimation technique (cf. [8]) that enable us to pass to limit and show the existence of global weak solutions to the original system (1.1)–(1.2). Finally, we get uniform-in-time L^r ($r \in [1, +\infty]$) estimates for the global solutions under more general assumptions by extending the methods in [10, 24] and show the long-time behavior.

The remaining part of the paper is organized as follows. In Section 2, we prove the well-posedness of an approximate problem and obtain some uniform estimates that are independent of the approximate parameter. In Section 3, we prove the existence of global

solutions to the original problem (1.1)–(1.2) by passing to the limit and show the uniqueness of the solution. In Section 4, we obtain some uniform-in-time estimates of the solutions and show that as time tends to infinity the global solutions will converge to a unique steady state.

2 Well-posedness of the Approximate System

In this paper, we use C , $C_i (i \in \mathbb{N})$ to denote generic constants that may vary in different places (even in the same estimate). Particular dependence of those constants will be explained in the text if necessary. $H^m(\mathbb{R}^3)$ ($m \in \mathbb{N}$) is used to denote the Sobolev space $W^{m,2}(\mathbb{R}^3)$, and $\|\cdot\|_{H^m(\mathbb{R}^3)}$ is the corresponding norm. We denote $L^r(\mathbb{R}^3)$ ($r \geq 1$) with norm $\|\cdot\|_{L^r(\mathbb{R}^3)}$. Moreover, we denote the vector space $\mathbf{L}^r(\mathbb{R}^3) = (L^r(\mathbb{R}^3))^3$ ($r \geq 1$). The norm on $\mathbf{L}^r(\mathbb{R}^3)$ is denoted by $\|\cdot\|_{\mathbf{L}^r(\mathbb{R}^3)}$. For any Hilbert space H , we denote its subspace

$$H_+ = \{f(x) \in H \mid f(x) \geq 0, \text{ a.e. } x \in \mathbb{R}^3\}.$$

To overcome the difficulty brought by the higher-order nonlinearity R , we introduce the following approximate problem. For any $\sigma > 0$, consider

$$(AP) \begin{cases} \partial_t n_\sigma = \operatorname{div}(\nabla n_\sigma + n_\sigma \nabla(\psi_\sigma + V_n)) - \tilde{R}(n_\sigma, p_\sigma, x), \\ \partial_t p_\sigma = \operatorname{div}(\nabla p_\sigma + p_\sigma \nabla(-\psi_\sigma + V_p)) - \tilde{R}(n_\sigma, p_\sigma, x), \\ -\Delta \psi_\sigma = n_\sigma - p_\sigma - D(x), \end{cases} \quad (2.1)$$

subject to the initial data

$$n_\sigma(x, t)|_{t=0} = n_I, \quad p_\sigma(x, t)|_{t=0} = p_I. \quad (2.2)$$

The approximated recombination-generation rate \tilde{R} in (2.1) is given by

$$\begin{aligned} \tilde{R}(n_\sigma, p_\sigma, x) &= R\left(\frac{n_\sigma}{1 + \sigma n_\sigma}, \frac{p_\sigma}{1 + \sigma p_\sigma}, x\right) \\ &= F\left(\frac{n_\sigma}{1 + \sigma n_\sigma}, \frac{p_\sigma}{1 + \sigma p_\sigma}\right) \left(\frac{n_\sigma}{1 + \sigma n_\sigma} \frac{p_\sigma}{1 + \sigma p_\sigma} - \mu_n \mu_p\right). \end{aligned} \quad (2.3)$$

Lemma 2.1. *Under assumptions (H2a)–(H2b), the function $\tilde{R} = \tilde{R}(n_\sigma, p_\sigma, x)$ satisfies the following properties*

(i) \tilde{R} has at most a linear growth for any $n_\sigma, p_\sigma \geq 0$, i.e.,

$$|\tilde{R}(n_\sigma, p_\sigma, x)| \leq C_\sigma(a(x) + n_\sigma + p_\sigma), \quad (2.4)$$

where $0 \leq a(x) \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $a(x) \in L^2(\mathbb{R}^3, e^{V_n(x)} dx) \cap L^2_+(\mathbb{R}^3, e^{V_p(x)} dx)$.

(ii) \tilde{R} is locally Lip-continuous in $L^2_+(\mathbb{R}^3, e^{V_i(x)} dx)$, $i = \{n, p\}$, such that for any $n_\sigma^{(1)}, n_\sigma^{(2)} \in L^2_+(\mathbb{R}^3, e^{V_n(x)} dx)$, $p_\sigma^{(1)}, p_\sigma^{(2)} \in L^2_+(\mathbb{R}^3, e^{V_p(x)} dx)$, there holds

$$\begin{aligned} &\|\tilde{R}(n_\sigma^{(1)}, p_\sigma^{(1)}, \cdot) - \tilde{R}(n_\sigma^{(2)}, p_\sigma^{(2)}, \cdot)\|_{L^2(\mathbb{R}^3, e^{V_i(x)} dx)} \\ &\leq \tilde{K} \left(\|n_\sigma^{(1)} - n_\sigma^{(2)}\|_{L^2(\mathbb{R}^3, e^{V_i(x)} dx)} + \|p_\sigma^{(1)} - p_\sigma^{(2)}\|_{L^2(\mathbb{R}^3, e^{V_i(x)} dx)} \right), \end{aligned} \quad (2.5)$$

where the constant \tilde{K} may depend on c_1, c_2, σ and V_b .

Proof. We observe that for any $\varphi \geq 0$, it holds

$$0 \leq \frac{\varphi}{1 + \sigma\varphi} \leq \frac{1}{\sigma} \quad \text{and} \quad \frac{\varphi}{1 + \sigma\varphi} \leq \varphi, \quad \forall \sigma > 0. \quad (2.6)$$

Due to this simple fact and assumptions (H2a)–(H2b), we can verify that

$$\begin{aligned} |\tilde{R}(n_\sigma, p_\sigma, x)| &\leq c_2 \left(1 + \frac{n_\sigma}{1 + \sigma n_\sigma} + \frac{p_\sigma}{1 + \sigma p_\sigma} \right) \left| \frac{n_\sigma}{1 + \sigma n_\sigma} \frac{p_\sigma}{1 + \sigma p_\sigma} - \mu_n \mu_p \right| \\ &\leq c_2 \left(1 + \frac{2}{\sigma} \right) \mu_n \mu_p + c_2 \left(\frac{1}{\sigma} + \frac{1}{\sigma^2} \right) (n_\sigma + p_\sigma). \end{aligned} \quad (2.7)$$

Then we can simply set $C_\sigma = c_2 \left(1 + \frac{2}{\sigma} + \frac{1}{\sigma^2} \right)$ and $a(x) = \mu_n \mu_p$, which obviously satisfies the required conditions by assumption (H1a).

For any $n_\sigma^{(1)}, n_\sigma^{(2)} \in L_+^2(\mathbb{R}^3, e^{V_n(x)} dx)$, $p_\sigma^{(1)}, p_\sigma^{(2)} \in L_+^2(\mathbb{R}^3, e^{V_p(x)} dx)$, we infer from (2.6) that

$$\left| \frac{n_\sigma^{(1)}}{1 + \sigma n_\sigma^{(1)}} - \frac{n_\sigma^{(2)}}{1 + \sigma n_\sigma^{(2)}} \right| \leq |n_\sigma^{(1)} - n_\sigma^{(2)}|, \quad \left| \frac{p_\sigma^{(1)}}{1 + \sigma p_\sigma^{(1)}} - \frac{p_\sigma^{(2)}}{1 + \sigma p_\sigma^{(2)}} \right| \leq |p_\sigma^{(1)} - p_\sigma^{(2)}|, \quad (2.8)$$

and as a result,

$$\begin{aligned} &\left| \frac{n_\sigma^{(1)}}{1 + \sigma n_\sigma^{(1)}} \frac{p_\sigma^{(1)}}{1 + \sigma p_\sigma^{(1)}} - \frac{n_\sigma^{(2)}}{1 + \sigma n_\sigma^{(2)}} \frac{p_\sigma^{(2)}}{1 + \sigma p_\sigma^{(2)}} \right| \\ &\leq \frac{p_\sigma^{(1)}}{1 + \sigma p_\sigma^{(1)}} \left| \frac{n_\sigma^{(1)}}{1 + \sigma n_\sigma^{(1)}} - \frac{n_\sigma^{(2)}}{1 + \sigma n_\sigma^{(2)}} \right| + \frac{n_\sigma^{(2)}}{1 + \sigma n_\sigma^{(2)}} \left| \frac{p_\sigma^{(1)}}{1 + \sigma p_\sigma^{(1)}} - \frac{p_\sigma^{(2)}}{1 + \sigma p_\sigma^{(2)}} \right| \\ &\leq \frac{1}{\sigma} (|n_\sigma^{(1)} - n_\sigma^{(2)}| + |p_\sigma^{(1)} - p_\sigma^{(2)}|). \end{aligned} \quad (2.9)$$

Denote

$$F_j = F \left(\frac{n_\sigma^{(j)}}{1 + \sigma n_\sigma^{(j)}}, \frac{p_\sigma^{(j)}}{1 + \sigma p_\sigma^{(j)}} \right), \quad j = 1, 2.$$

Then we get

$$\begin{aligned} &\|\tilde{R}(n_\sigma^{(1)}, p_\sigma^{(1)}, \cdot) - \tilde{R}(n_\sigma^{(2)}, p_\sigma^{(2)}, \cdot)\|_{L^2(\mathbb{R}^3, e^{V_i(x)} dx)} \\ &\leq \|(F_1 - F_2) \mu_n \mu_p\|_{L^2(\mathbb{R}^3, e^{V_i(x)} dx)} \\ &\quad + \left\| F_1 \left(\frac{n_\sigma^{(1)}}{1 + \sigma n_\sigma^{(1)}} \frac{p_\sigma^{(1)}}{1 + \sigma p_\sigma^{(1)}} - \frac{n_\sigma^{(2)}}{1 + \sigma n_\sigma^{(2)}} \frac{p_\sigma^{(2)}}{1 + \sigma p_\sigma^{(2)}} \right) \right\|_{L^2(\mathbb{R}^3, e^{V_i(x)} dx)} \\ &\quad + \left\| (F_1 - F_2) \frac{n_\sigma^{(2)}}{1 + \sigma n_\sigma^{(2)}} \frac{p_\sigma^{(2)}}{1 + \sigma p_\sigma^{(2)}} \right\|_{L^2(\mathbb{R}^3, e^{V_i(x)} dx)} \\ &:= I_1 + I_2 + I_3, \quad i = \{n, p\} \end{aligned} \quad (2.10)$$

For the case $i = n$, it follows from (2.6), (2.8), (2.9), (H2a)–(H2b) and Remark 1.1 that

$$I_1 \leq \left\| c_1 \left(\left| \frac{n_\sigma^{(1)}}{1 + \sigma n_\sigma^{(1)}} - \frac{n_\sigma^{(2)}}{1 + \sigma n_\sigma^{(2)}} \right| + \left| \frac{p_\sigma^{(1)}}{1 + \sigma p_\sigma^{(1)}} - \frac{p_\sigma^{(2)}}{1 + \sigma p_\sigma^{(2)}} \right| \right) \mu_n \mu_p \right\|_{L^2(\mathbb{R}^3, e^{V_n(x)} dx)}$$

$$\begin{aligned}
&\leq \|c_1(|n_\sigma^{(1)} - n_\sigma^{(2)}| + |p_\sigma^{(1)} - p_\sigma^{(2)}|)\mu_n\mu_p\|_{L^2(\mathbb{R}^3, e^{V_n(x)}dx)} \\
&\leq 2c_1 e^{2V_b} \left(\|n_\sigma^{(1)} - n_\sigma^{(2)}\|_{L^2(\mathbb{R}^3, e^{V_n(x)}dx)} + \|p_\sigma^{(1)} - p_\sigma^{(2)}\|_{L^2(\mathbb{R}^3, e^{V_n(x)}dx)} \right), \\
I_2 &\leq \left\| \frac{c_2}{\sigma} \left(1 + \frac{n_\sigma^{(1)}}{1 + \sigma n_\sigma^{(1)}} + \frac{p_\sigma^{(1)}}{1 + \sigma p_\sigma^{(1)}} \right) (|n_\sigma^{(1)} - n_\sigma^{(2)}| + |p_\sigma^{(1)} - p_\sigma^{(2)}|) \right\|_{L^2(\mathbb{R}^3, e^{V_n(x)}dx)} \\
&\leq \frac{2c_2}{\sigma} \left(1 + \frac{2}{\sigma} \right) \left(\|n_\sigma^{(1)} - n_\sigma^{(2)}\|_{L^2(\mathbb{R}^3, e^{V_n(x)}dx)} + \|p_\sigma^{(1)} - p_\sigma^{(2)}\|_{L^2(\mathbb{R}^3, e^{V_n(x)}dx)} \right), \\
I_3 &\leq \frac{1}{\sigma^2} \|c_1(|n_\sigma^{(1)} - n_\sigma^{(2)}| + |p_\sigma^{(1)} - p_\sigma^{(2)}|)\|_{L^2(\mathbb{R}^3, e^{V_n(x)}dx)} \\
&\leq \frac{2c_1}{\sigma^2} \left(\|n_\sigma^{(1)} - n_\sigma^{(2)}\|_{L^2(\mathbb{R}^3, e^{V_n(x)}dx)} + \|p_\sigma^{(1)} - p_\sigma^{(2)}\|_{L^2(\mathbb{R}^3, e^{V_n(x)}dx)} \right).
\end{aligned}$$

Collecting the above estimates together, we see that (2.5) ($i = n$) holds with

$$\tilde{K} = 2c_1 e^{2V_b} + \frac{2c_2}{\sigma} \left(1 + \frac{2}{\sigma} \right) + \frac{2c_1}{\sigma^2}.$$

The case $i = p$ can be treated in the same way. The proof is complete. \square

Now we state the main result of this section.

Theorem 2.1. *Suppose that assumptions (H1a)–(H3) are satisfied. For any $\sigma > 0$, $n_I \in L^2(\mathbb{R}^3, e^{V_n(x)}dx) \cap L^4(\mathbb{R}^3)$, $p_I \in L^2(\mathbb{R}^3, e^{V_p(x)}dx) \cap L^4(\mathbb{R}^3)$, $n_I, p_I \geq 0$, problem (2.1)–(2.2) admits a unique global weak solution $(n_\sigma, p_\sigma, \psi_\sigma)$ such that for any $T > 0$,*

$$\begin{aligned}
n_\sigma &\in C([0, T]; L^2(\mathbb{R}^3, e^{V_n(x)}dx)), \quad \nabla n_\sigma \in L^2((0, T); \mathbf{L}^2(\mathbb{R}^3, e^{V_n(x)}dx)), \\
p_\sigma &\in C([0, T]; L^2(\mathbb{R}^3, e^{V_p(x)}dx)), \quad \nabla p_\sigma \in L^2((0, T); \mathbf{L}^2(\mathbb{R}^3, e^{V_p(x)}dx)), \\
n_\sigma(t) &\geq 0, \quad p_\sigma(t) \geq 0, \quad t \in [0, T], \quad a.e. \ x \in \mathbb{R}^3.
\end{aligned}$$

$\psi_\sigma = \psi_\sigma(t, x)$ is the Newtonian potential with respect to $n_\sigma(t, x), p_\sigma(t, x)$ given by

$$\psi_\sigma(t, x) = \frac{1}{S_3} \int_{\mathbb{R}^3} \frac{n_\sigma(t, y) - p_\sigma(t, y) - D(y)}{|x - y|} dy, \quad \text{with } S_3 = \frac{2\pi^{\frac{3}{2}}}{\Gamma(\frac{3}{2})}.$$

The proof of Theorem 2.1 consists of several steps. We introduce the following transformation of unknown variables (cf. [2, 24])

$$u := n_\sigma e^{\frac{V_n(x)}{2}}, \quad v := p_\sigma e^{\frac{V_p(x)}{2}}. \quad (2.11)$$

A direct computation shows that u and v satisfy the following transformed approximate system

$$(TAP) \begin{cases} u_t - \Delta u + A_n(x)u = f_1(u, v, \psi_\sigma), \\ v_t - \Delta v + A_p(x)v = f_2(u, v, \psi_\sigma), \\ -\Delta \psi_\sigma = ue^{-\frac{V_n(x)}{2}} - ve^{-\frac{V_p(x)}{2}} - D(x), \end{cases} \quad (2.12)$$

where

$$A_n(x) = \frac{1}{4}|\nabla V_n(x)|^2 - \frac{1}{2}\Delta V_n(x) + K, \quad A_p(x) = \frac{1}{4}|\nabla V_p(x)|^2 - \frac{1}{2}\Delta V_p(x) + K.$$

System (2.12) is subject to the initial data

$$u(x, t)|_{t=0} = n_I(x)e^{\frac{V_n(x)}{2}} := u_I, \quad v(x, t)|_{t=0} = p_I(x)e^{\frac{V_p(x)}{2}} := v_I. \quad (2.13)$$

It follows from (1.6) that $A_n(x)$ and $A_p(x)$ are bounded from below, i.e.

$$A_n(x) \geq \frac{K}{2}, \quad A_p(x) \geq \frac{K}{2}, \quad \text{a.e. } x \in \mathbb{R}^3.$$

Under the transformation (2.11), the right-hand side of (2.12) are given by:

$$\begin{aligned} f_1(u, v, \psi_\sigma) &= Ku + e^{\frac{V_n(x)}{2}} (\operatorname{div}(n_\sigma \nabla \psi_\sigma) - \tilde{R}(n_\sigma, p_\sigma)) \\ &= Ku + \nabla u \cdot \nabla \psi_\sigma - \frac{1}{2}u \nabla \psi_\sigma \cdot \nabla V_n - u^2 e^{-\frac{V_n(x)}{2}} + uve^{-\frac{V_p(x)}{2}} + D(x)u \\ &\quad - e^{\frac{V_n(x)}{2}} \tilde{R}(n_\sigma, p_\sigma, x), \\ f_2(u, v, \psi_\sigma) &= Kv + e^{\frac{V_p(x)}{2}} (\operatorname{div}(-p_\sigma \nabla \psi_\sigma) - \tilde{R}(n_\sigma, p_\sigma)) \\ &= Kv - \nabla v \cdot \nabla \psi_\sigma + \frac{1}{2}v \nabla \psi_\sigma \cdot \nabla V_p - v^2 e^{-\frac{V_p(x)}{2}} + uve^{-\frac{V_n(x)}{2}} - D(x)v \\ &\quad - e^{\frac{V_p(x)}{2}} \tilde{R}(n_\sigma, p_\sigma, x). \end{aligned}$$

As a first step, we prove the local well-posedness of the transformed approximate problem (2.12)–(2.13).

Proposition 2.1. *Suppose that (H1a)–(H3) are satisfied and $u_I, v_I \in L_+^2(\mathbb{R}^3)$. Then for any $\sigma > 0$, there exists $T_\sigma > 0$ such that problem (2.12)–(2.13) admits a unique solution (u, v, ψ_σ) on $[0, T_\sigma]$, which satisfies*

$$\begin{aligned} u, v &\in C([0, T_\sigma]; L_+^2(\mathbb{R}^3)), \quad \nabla u, \nabla v \in L^2(0, T_\sigma; \mathbf{L}^2(\mathbb{R}^3)), \\ u \nabla V_n, v \nabla V_p &\in L^2(0, T_\sigma; \mathbf{L}^2(\mathbb{R}^3)). \end{aligned}$$

The potential ψ_σ is given by

$$\psi_\sigma(t, x) = \frac{1}{S_3} \int_{\mathbb{R}^3} \frac{n_\sigma(t, y) - p_\sigma(t, y) - D(y)}{|x - y|} dy.$$

Proof. We consider the following auxiliary linear problem of the transformed approximated problem (2.12)–(2.13), such that for any $\tilde{u}, \tilde{v} \in C([0, T]; L^2(\mathbb{R}^3))$, $\nabla \tilde{u}, \nabla \tilde{v} \in L^2(0, T; \mathbf{L}^2(\mathbb{R}^3))$, $\tilde{u} \nabla V_n, \tilde{v} \nabla V_p \in L^2(0, T; \mathbf{L}^2(\mathbb{R}^3))$,

$$(ATAP) \begin{cases} u_t - \Delta u + A_n(x)u = f_1^+(\tilde{u}, \tilde{v}, \tilde{\psi}_\sigma), \\ v_t - \Delta v + A_p(x)v = f_2^+(\tilde{u}, \tilde{v}, \tilde{\psi}_\sigma), \\ u(x, t)|_{t=0} = u_I, \quad v(x, t)|_{t=0} = v_I, \end{cases} \quad (2.14)$$

where $\tilde{\psi}_\sigma$ satisfies

$$-\Delta \tilde{\psi}_\sigma = \tilde{u}e^{-\frac{V_n(x)}{2}} - \tilde{v}e^{-\frac{V_p(x)}{2}} - D(x), \quad (2.15)$$

and the nonlinearities f_1^+, f_2^+ are given by

$$\begin{aligned} f_1^+(\tilde{u}, \tilde{v}, \tilde{\psi}_\sigma) &= K\tilde{u}^+ + e^{\frac{V_n(x)}{2}} \left[\operatorname{div} \left(\tilde{u}^+ e^{-\frac{V_n(x)}{2}} \nabla \tilde{\psi}_\sigma \right) - \tilde{R} \left(\tilde{u}^+ e^{-\frac{V_n(x)}{2}}, \tilde{v}^+ e^{-\frac{V_p(x)}{2}}, x \right) \right], \\ f_2^+(\tilde{u}, \tilde{v}, \tilde{\psi}_\sigma) &= K\tilde{v}^+ + e^{\frac{V_p(x)}{2}} \left[-\operatorname{div} \left(\tilde{v}^+ e^{-\frac{V_p(x)}{2}} \nabla \tilde{\psi}_\sigma \right) - \tilde{R} \left(\tilde{u}^+ e^{-\frac{V_n(x)}{2}}, \tilde{v}^+ e^{-\frac{V_p(x)}{2}}, x \right) \right], \\ \text{with } \tilde{u}^+ &:= \max\{0, \tilde{u}\}, \quad \tilde{v}^+ := \max\{0, \tilde{v}\}. \end{aligned}$$

Since now the nonlinearity \tilde{R} in the approximate problem (2.1) satisfies the properties in Lemma 2.1, using assumptions (H1a)–(H3) we are able to prove the local well-posedness of problem (2.12)–(2.13) by the contraction mapping principle as in [24, Theorem 2.2]. Since the proof is the same, we only sketch it here. Denote

$$\begin{aligned} \Sigma_T &= \left\{ (u, v) \in C([0, T]; L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)), \right. \\ &\quad (\nabla u, \nabla v) \in L^2(0, T; \mathbf{L}^2(\mathbb{R}^3) \times \mathbf{L}^2(\mathbb{R}^3)) : \\ &\quad u(0) = u_I \geq 0, \quad v(0) = v_I \geq 0, \quad \max_{0 \leq t \leq T} \left(\|u\|_{L^2(\mathbb{R}^3)}^2 + \|v\|_{L^2(\mathbb{R}^3)}^2 \right) \leq 2M, \\ &\quad \int_0^T \left(\|\nabla u(t)\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 + \|\nabla v(t)\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 \right) dt \leq 2M, \\ &\quad \left. \int_0^T \int_{\mathbb{R}^3} (|u(t) \nabla V_n|^2 + |v(t) \nabla V_p|^2) dt \leq 2M. \right\} \end{aligned}$$

where

$$M := \|u_I\|_{L^2(\mathbb{R}^3)}^2 + \|v_I\|_{L^2(\mathbb{R}^3)}^2.$$

Then we can prove that there exists a sufficiently small $T_\sigma > 0$ such that the mapping $\mathcal{G} : (\tilde{u}, \tilde{v}) \mapsto (u, v)$ defined by (2.14) maps Σ_{T_σ} to itself and is a strict contraction. Hence, the contraction principle entails that \mathcal{G} has a unique fixed point in Σ_{T_σ} such that $\mathcal{G}(u, v) = (u, v)$. Next, due to the special structure of the approximated recombination–generation rate \tilde{R} , using the idea in [13], one can show the nonnegativity of the fixed point (u, v) of \mathcal{G} , provided that the initial data u_I, v_I are nonnegative (cf. [24, Theorem 2.2]). Since $\tilde{u}^+ = \tilde{u}$, $\tilde{v}^+ = \tilde{v}$ if $\tilde{u}, \tilde{v} \geq 0$, we see that $f_i^+(\tilde{u}, \tilde{v}, \tilde{\psi}_\sigma) = f_i(\tilde{u}, \tilde{v}, \tilde{\psi}_\sigma)$ ($i = 1, 2$) for $\tilde{u}, \tilde{v} \geq 0$. Thus, (u, v) is the local solution of problem (2.12)–(2.13). The details are omitted here. \square

Lemma 2.2. *Assume that (H1a)–(H3) are satisfied. For any $T > 0$, if $n_I, p_I \in L^r(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, $r \in \mathbb{N}$, we have*

$$\|n_\sigma(t)\|_{L^s(\mathbb{R}^3)} + \|p_\sigma(t)\|_{L^s(\mathbb{R}^3)} \leq C_T, \quad 0 \leq t \leq T, \quad s = 1, \dots, r. \quad (2.16)$$

Moreover,

$$\|\nabla \psi_\sigma\|_{L^\infty(\mathbb{R}^3)} \leq C_T, \quad 0 \leq t \leq T, \quad (2.17)$$

provided that $n_I, p_I \in L^4(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. In particular, the constant C_T is independent of $\sigma > 0$.

Proof. Integrating the equations for n_σ and p_σ in (2.1) on \mathbb{R}^3 , we infer from (2.6), (H2a)–(H2b) that

$$\begin{aligned}
& \frac{d}{dt}(\|n_\sigma\|_{L^1(\mathbb{R}^3)} + \|p_\sigma\|_{L^1(\mathbb{R}^3)}) = -2 \int_{\mathbb{R}^3} R\left(\frac{n_\sigma}{1+\sigma n_\sigma}, \frac{p_\sigma}{1+\sigma p_\sigma}, x\right) dx \\
& = -2 \int_{\mathbb{R}^3} \left(\frac{n_\sigma p_\sigma}{(1+\sigma n_\sigma)(1+\sigma p_\sigma)} - \mu_n \mu_p \right) F\left(\frac{n_\sigma}{1+\sigma n_\sigma}, \frac{p_\sigma}{1+\sigma p_\sigma}\right) dx \\
& \leq 2 \int_{\mathbb{R}^3} \mu_n \mu_p F\left(\frac{n_\sigma}{1+\sigma n_\sigma}, \frac{p_\sigma}{1+\sigma p_\sigma}\right) dx \\
& \leq 2c_2 \int_{\mathbb{R}^3} \mu_n \mu_p (1+n_\sigma+p_\sigma) dx \\
& \leq C(1 + \|n_\sigma\|_{L^1(\mathbb{R}^3)} + \|p_\sigma\|_{L^1(\mathbb{R}^3)}),
\end{aligned} \tag{2.18}$$

which yields

$$\|n_\sigma(t)\|_{L^1(\mathbb{R}^3)} + \|p_\sigma(t)\|_{L^1(\mathbb{R}^3)} \leq (\|n_I\|_{L^1(\mathbb{R}^3)} + \|p_I\|_{L^1(\mathbb{R}^3)} + 1)e^{CT}, \quad \forall t \in [0, T]. \tag{2.19}$$

Next, multiplying the equations for n_σ and p_σ by n_σ^r, p_σ^r ($r \in \mathbb{N}$) respectively, integrating on \mathbb{R}^3 and adding the resultants, we obtain that

$$\begin{aligned}
& \frac{1}{r+1} \frac{d}{dt} \int_{\mathbb{R}^3} (n_\sigma^{r+1} + p_\sigma^{r+1}) dx + \frac{4r}{(r+1)^2} \int_{\mathbb{R}^3} \left(\left| \nabla \left(n_\sigma^{\frac{r+1}{2}} \right) \right|^2 + \left| \nabla \left(p_\sigma^{\frac{r+1}{2}} \right) \right|^2 \right) dx \\
& = \frac{r}{r+1} \int_{\mathbb{R}^3} \Delta \psi_\sigma (n_\sigma^{r+1} - p_\sigma^{r+1}) + \frac{r}{r+1} \int_{\mathbb{R}^3} (\Delta V_n n_\sigma^{r+1} + \Delta V_p p_\sigma^{r+1}) dx \\
& \quad - \int_{\mathbb{R}^3} R\left(\frac{n_\sigma}{1+\sigma n_\sigma}, \frac{p_\sigma}{1+\sigma p_\sigma}, x\right) (n_\sigma^r + p_\sigma^r) dx.
\end{aligned} \tag{2.20}$$

Since $n_\sigma, p_\sigma \geq 0$, we observe that

$$\begin{aligned}
& - \int_{\mathbb{R}^3} R\left(\frac{n_\sigma}{1+\sigma n_\sigma}, \frac{p_\sigma}{1+\sigma p_\sigma}, x\right) (n_\sigma^r + p_\sigma^r) dx \\
& = - \int_{\mathbb{R}^3} \frac{n_\sigma p_\sigma}{(1+\sigma n_\sigma)(1+\sigma p_\sigma)} F\left(\frac{n_\sigma}{1+\sigma n_\sigma}, \frac{p_\sigma}{1+\sigma p_\sigma}\right) (n_\sigma^r + p_\sigma^r) dx \\
& \quad + \int_{\mathbb{R}^3} \mu_n \mu_p F\left(\frac{n_\sigma}{1+\sigma n_\sigma}, \frac{p_\sigma}{1+\sigma p_\sigma}\right) (n_\sigma^r + p_\sigma^r) dx \\
& \leq C \int_{\mathbb{R}^3} (n_\sigma^r + p_\sigma^r + n_\sigma^{r+1} + p_\sigma^{r+1}) dx.
\end{aligned} \tag{2.21}$$

On the other hand, from the Poisson equation and the elementary calculation

$$(a^{r+1} - b^{r+1})(b - a) = - \sum_{k=0}^r a^{r-1} b^k (a - b)^2 \leq 0, \quad \forall a, b \geq 0, \tag{2.22}$$

we infer from (H3) that

$$\begin{aligned}
\frac{r}{r+1} \int_{\mathbb{R}^3} \Delta \psi_\sigma (n_\sigma^{r+1} - p_\sigma^{r+1}) dx & = \frac{r}{r+1} \int_{\mathbb{R}^3} (p_\sigma - n_\sigma + D(x)) (n_\sigma^{r+1} - p_\sigma^{r+1}) dx \\
& \leq \|D(x)\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} (n_\sigma^{r+1} + p_\sigma^{r+1}) dx.
\end{aligned} \tag{2.23}$$

Besides, it follows from (H1a) that

$$\frac{r}{r+1} \int_{\mathbb{R}^3} (\Delta V_n n_\sigma^{r+1} + \Delta V_p p_\sigma^{r+1}) dx \leq K \int_{\mathbb{R}^3} (n_\sigma^{r+1} + p_\sigma^{r+1}) dx. \quad (2.24)$$

Summing up, we have

$$\begin{aligned} & \frac{1}{r+1} \frac{d}{dt} \int_{\mathbb{R}^3} (n_\sigma^{r+1} + p_\sigma^{r+1}) dx + \frac{4r}{(r+1)^2} \int_{\mathbb{R}^3} \left(\left| \nabla \left(n_\sigma^{\frac{r+1}{2}} \right) \right|^2 + \left| \nabla \left(p_\sigma^{\frac{r+1}{2}} \right) \right|^2 \right) dx \\ & \leq C \int_{\mathbb{R}^3} (n_\sigma^r + p_\sigma^r + n_\sigma^{r+1} + p_\sigma^{r+1}) dx \\ & \leq C \int_{\mathbb{R}^3} (1 + n_\sigma^{r+1} + p_\sigma^{r+1}) dx. \end{aligned} \quad (2.25)$$

Then it follows from the Gronwall inequality that

$$\|n_\sigma\|_{L^{r+1}(\mathbb{R}^3)}^{r+1} + \|p_\sigma\|_{L^{r+1}(\mathbb{R}^3)}^{r+1} + \int_0^T \int_{\mathbb{R}^3} \left(\left| \nabla \left(n_\sigma^{\frac{r+1}{2}} \right) \right|^2 + \left| \nabla \left(p_\sigma^{\frac{r+1}{2}} \right) \right|^2 \right) dx dt \leq C_T, \quad r \in \mathbb{N}, \quad (2.26)$$

provided that $\int_{\mathbb{R}^3} (n_I^{r+1} + p_I^{r+1}) dx < \infty$.

Since

$$\psi_\sigma = \frac{1}{S_3} \int_{\mathbb{R}^3} \frac{n_\sigma(t, y) - p_\sigma(t, y) - D(y)}{|x - y|} dy, \quad (2.27)$$

we have

$$\nabla \psi_\sigma(t, x) = \frac{1}{S_3} \int_{\mathbb{R}^3} \frac{(n_\sigma(t, y) - p_\sigma(t, y) - D(y))(x - y)}{|x - y|^3} dy. \quad (2.28)$$

For any $x \in \mathbb{R}^3$,

$$|\nabla \psi_\sigma(t, x)| \leq C \int_{\mathbb{R}^3} \frac{|n_\sigma(t, y) - p_\sigma(t, y) - D(y)|}{|x - y|^2} dy. \quad (2.29)$$

It follows from [19, Corollary 2.2] (a direct consequence of the Hardy–Littlewood–Sobolev inequality [27]) that for any $q, q' \in (1, +\infty)$ with $\frac{1}{q} = \frac{1}{q'} - \frac{1}{3}$,

$$\|\nabla \psi_\sigma\|_{\mathbf{L}^q(\mathbb{R}^3)} \leq C \|n_\sigma - p_\sigma - D(x)\|_{L^{q'}(\mathbb{R}^3)}. \quad (2.30)$$

Besides, if $n_\sigma, p_\sigma \in L^\infty(0, T; L^{q'}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3))$ with $q' > 3$, then we have

$$\nabla \psi_\sigma \in L^\infty(0, T; \mathbf{L}^\infty(\mathbb{R}^3)). \quad (2.31)$$

The proof is complete. \square

Based on the *a priori* estimates obtained in Lemma 2.2, we can prove existence of global solutions to problem (2.1)–(2.2).

Proposition 2.2. *Suppose that all assumptions in Proposition 2.1 are satisfied. Assume in addition that $n_I, p_I \in L^4(\mathbb{R}^3)$. The local solution $(n_\sigma, p_\sigma, \psi_\sigma)$ obtained in Proposition 2.1 is global.*

Proof. Multiplying the first two equations in (2.12) by u and v respectively, integrating on \mathbb{R}^3 , we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} A_n(x) u^2 dx = \int_{\mathbb{R}^3} f_1(u, v, \psi_\sigma) u dx \\ &= K \int_{\mathbb{R}^3} u^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} u^2 \Delta \psi_\sigma dx - \frac{1}{2} \int_{\mathbb{R}^3} u^2 \nabla \psi_\sigma \cdot \nabla V_n dx \\ & \quad - \int_{\mathbb{R}^3} u e^{\frac{V_n(x)}{2}} R \left(\frac{u e^{-\frac{V_n(x)}{2}}}{1 + \sigma u e^{-\frac{V_n(x)}{2}}}, \frac{v e^{-\frac{V_p(x)}{2}}}{1 + \sigma v e^{-\frac{V_p(x)}{2}}}, x \right) dx, \end{aligned} \quad (2.32)$$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|v\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla v\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} A_p(x) v^2 dx = \int_{\mathbb{R}^3} f_2(u, v, \psi_\sigma) v dx \\ &= K \int_{\mathbb{R}^3} v^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} v^2 \Delta \psi_\sigma dx + \frac{1}{2} \int_{\mathbb{R}^3} v^2 \nabla \psi_\sigma \cdot \nabla V_p dx \\ & \quad - \int_{\mathbb{R}^3} v e^{\frac{V_p(x)}{2}} R \left(\frac{u e^{-\frac{V_n(x)}{2}}}{1 + \sigma u e^{-\frac{V_n(x)}{2}}}, \frac{v e^{-\frac{V_p(x)}{2}}}{1 + \sigma v e^{-\frac{V_p(x)}{2}}}, x \right) dx. \end{aligned} \quad (2.33)$$

From the Poisson equation for ψ_σ , Lemma 2.2 and the Gagliardo–Nirenberg inequality $\|u\|_{L^{\frac{8}{3}}(\mathbb{R}^3)} \leq C \|\nabla u\|_{\mathbf{L}^2(\mathbb{R}^3)}^{\frac{3}{8}} \|u\|_{L^2(\mathbb{R}^3)}^{\frac{5}{8}}$, we have

$$\begin{aligned} & \int_{\mathbb{R}^3} (u^2 - v^2) \Delta \psi_\sigma dx = \int_{\mathbb{R}^3} (u^2 - v^2) (-n_\sigma + p_\sigma) dx + \int_{\mathbb{R}^3} D(x) (u^2 - v^2) dx \\ & \leq \left(\|u\|_{L^{\frac{8}{3}}(\mathbb{R}^3)}^2 + \|p\|_{L^{\frac{8}{3}}(\mathbb{R}^3)}^2 \right) (\|n_\sigma\|_{L^4(\mathbb{R}^3)} + \|p_\sigma\|_{L^4(\mathbb{R}^3)}) \\ & \quad + \|D(x)\|_{L^\infty(\mathbb{R}^3)} \left(\|u\|_{L^2(\mathbb{R}^3)}^2 + \|v\|_{L^2(\mathbb{R}^3)}^2 \right) \\ & \leq \frac{1}{2} \left(\|\nabla u\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 + \|\nabla v\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 \right) + C \left(\|u\|_{L^2(\mathbb{R}^3)}^2 + \|v\|_{L^2(\mathbb{R}^3)}^2 \right). \end{aligned} \quad (2.34)$$

We infer from the assumption $u_I, v_I \in L_+^2(\mathbb{R}^3)$ and the Cauchy–Schwarz inequality that

$$\int_{\mathbb{R}^3} n_I dx \leq \frac{1}{2} \int_{\mathbb{R}^3} \mu_n dx + \frac{1}{2} \int_{\mathbb{R}^3} u_I^2 dx < +\infty, \quad \int_{\mathbb{R}^3} p_I dx \leq \frac{1}{2} \int_{\mathbb{R}^3} \mu_p dx + \frac{1}{2} \int_{\mathbb{R}^3} v_I^2 dx < +\infty, \quad (2.35)$$

namely, $n_I, p_I \in L^1(\mathbb{R}^3)$. Lemma 2.2 implies that if the initial data satisfy $n_I, p_I \in L^4(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, then the estimate (2.17) holds. As a consequence, we get

$$\begin{aligned} \left| \int_{\mathbb{R}^3} u^2 \nabla \psi_\sigma \cdot \nabla V_n dx \right| & \leq \|\nabla \psi_\sigma\|_{\mathbf{L}^\infty(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^3)} \left(\int_{\mathbb{R}^3} u^2 |\nabla V_n|^2 dx \right)^{\frac{1}{2}} \\ & \leq \frac{1}{2} \int_{\mathbb{R}^3} u^2 |\nabla V_n|^2 dx + C_T \|u\|_{L^2(\mathbb{R}^3)}^2, \end{aligned} \quad (2.36)$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}^3} v^2 \nabla \psi_\sigma \cdot \nabla V_p dx \right| & \leq \|\nabla \psi_\sigma\|_{\mathbf{L}^\infty(\mathbb{R}^3)} \|v\|_{L^2(\mathbb{R}^3)} \left(\int_{\mathbb{R}^3} v^2 |\nabla V_p|^2 dx \right)^{\frac{1}{2}} \\ & \leq \frac{1}{2} \int_{\mathbb{R}^3} v^2 |\nabla V_p|^2 dx + C_T \|v\|_{L^2(\mathbb{R}^3)}^2. \end{aligned} \quad (2.37)$$

It follows from the nonnegativity of (u, v) and assumptions (H1b), (H2b) that

$$\begin{aligned}
& - \int_{\mathbb{R}^3} \left(ue^{\frac{V_n(x)}{2}} + ve^{\frac{V_p(x)}{2}} \right) R \left(\frac{ue^{-\frac{V_n(x)}{2}}}{1 + \sigma ue^{-\frac{V_n(x)}{2}}}, \frac{ve^{-\frac{V_p(x)}{2}}}{1 + \sigma ve^{-\frac{V_p(x)}{2}}}, x \right) dx \\
& \leq \int_{\mathbb{R}^3} \left(ue^{\frac{V_n(x)}{2}} + ve^{\frac{V_p(x)}{2}} \right) F \left(\frac{ue^{-\frac{V_n(x)}{2}}}{1 + \sigma ue^{-\frac{V_n(x)}{2}}}, \frac{ve^{-\frac{V_p(x)}{2}}}{1 + \sigma ve^{-\frac{V_p(x)}{2}}}, x \right) \mu_n \mu_p dx \\
& \leq C \int_{\mathbb{R}^3} \left(ue^{\frac{V_n(x)}{2}} + ve^{\frac{V_p(x)}{2}} \right) \left(\mu_n \mu_p + ue^{-\frac{V_n(x)}{2}} + ve^{-\frac{V_p(x)}{2}} \right) dx \\
& \leq C \left(1 + \|u\|_{L^2(\mathbb{R}^3)}^2 + \|v\|_{L^2(\mathbb{R}^3)}^2 \right). \tag{2.38}
\end{aligned}$$

As a result, we have

$$\begin{aligned}
& \frac{d}{dt} (\|u\|_{L^2(\mathbb{R}^3)}^2 + \|v\|_{L^2(\mathbb{R}^3)}^2) + \|\nabla u\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 + \|\nabla v\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 \\
& + \int_{\mathbb{R}^3} u^2 |\nabla V_n|^2 dx + \int_{\mathbb{R}^3} v^2 |\nabla V_p|^2 dx \\
& \leq C_T (1 + \|u\|_{L^2(\mathbb{R}^3)}^2 + \|v\|_{L^2(\mathbb{R}^3)}^2). \tag{2.39}
\end{aligned}$$

It follows from the Gronwall inequality that for any $T > 0$ and $t \in [0, T]$,

$$\|u(t)\|_{L^2(\mathbb{R}^3)} + \|v(t)\|_{L^2(\mathbb{R}^3)} \leq C_T, \tag{2.40}$$

$$\int_0^T (\|\nabla u\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 + \|\nabla v\|_{\mathbf{L}^2(\mathbb{R}^3)}^2) dt \leq C_T, \tag{2.41}$$

$$\int_0^T \int_{\mathbb{R}^3} (u^2 |\nabla V_n|^2 dx + v^2 |\nabla V_p|^2 dx) dt \leq C_T, \tag{2.42}$$

where C_T is a constant depending on $T, c_2, V_b, \|u_I\|_{L^2(\mathbb{R}^3)}, \|v_I\|_{L^2(\mathbb{R}^3)}, \|n_I\|_{L^4(\mathbb{R}^3)}, \|p_I\|_{L^4(\mathbb{R}^3)}$ but it is independent of σ .

Then we are able to extend the local solution $(n_\sigma, p_\sigma, \psi_\sigma)$ obtained in Proposition 2.1 to the interval $[0, T]$ for arbitrary $T > 0$. The proof is complete. \square

Proof Theorem 2.1. Recalling the transformation (2.11), we obtain the conclusion in Theorem 2.1 from Propositions 2.1, 2.2.

3 Well-posedness of the Original Problem

In this section, we prove the existence and uniqueness of global solutions to the Cauchy problem of original system (1.1). For this purpose, we shall derive some *a priori* estimates on the solutions $(n_\sigma, p_\sigma, \psi_\sigma)$ of the approximate problem that are uniform in the parameter $\sigma > 0$. Then, we pass to the limit as $\sigma \rightarrow 0$ to achieve our goal. In Lemma 2.2, we have already shown the uniform estimates on $\|n_\sigma(t)\|_{L^r(\mathbb{R}^3)}, \|p_\sigma(t)\|_{L^r(\mathbb{R}^3)}$ on arbitrary interval $[0, T]$. Next, we proceed to obtain uniform estimates on the L^∞ norms of n_σ and p_σ via a Stampacchia-type L^∞ estimation technique (cf. [8]). The following technical lemma (see [8, Lemma 2.3]) plays an important role in the proof.

Lemma 3.1. Suppose $\omega(k)$ is a nonnegative non-increasing function on $[k_0, +\infty)$, and there are constants γ, β such that

$$\omega(\hat{k}) \leq M(k)(\hat{k} - k)^{-\gamma} \omega(k)^{1+\beta}, \quad \forall \hat{k} > k \geq k_0, \quad (3.1)$$

where the function $M(k)$ is non-decreasing and satisfies

$$0 \leq k^{-\gamma} M(k) \leq M_0, \quad \forall k \in [k_0, +\infty). \quad (3.2)$$

Then

$$\omega(k^*) = 0, \quad \text{with } k^* = 2k_0 \left(1 + 2^{\frac{1+2\beta}{\beta^2}} M_0^{\frac{1+\beta}{\beta\gamma}} \omega(k_0)^{\frac{1+\beta}{\gamma}} \right). \quad (3.3)$$

Lemma 3.2. Suppose that all assumptions in Theorem 2.1 are satisfied. Assume in addition that $n_I, p_I \in L^\infty(\mathbb{R}^3)$. Then for any $T > 0$, we have

$$\|n_\sigma(t)\|_{L^\infty(\mathbb{R}^3)} \leq C_T, \quad \|p_\sigma(t)\|_{L^\infty(\mathbb{R}^3)} \leq C_T, \quad 0 \leq t \leq T, \quad (3.4)$$

where the constant C_T is independent of $\sigma > 0$.

Proof. Denote

$$k_0 := \max\{\|n_I\|_{L^\infty(\mathbb{R}^3)}, \|p_I\|_{L^\infty(\mathbb{R}^3)}\} \geq 0. \quad (3.5)$$

For any $k \geq k_0$, we set

$$B_{nk}(t) = \{x \in \mathbb{R}^3, n_\sigma(t, x) > k\}, \quad B_{pk}(t) = \{x \in \mathbb{R}^3, p_\sigma(t, x) > k\}. \quad (3.6)$$

and for arbitrary $T > 0$,

$$\omega_T(k) = \sup_{0 \leq t \leq T} (|B_{nk}(t)| + |B_{pk}(t)|). \quad (3.7)$$

It is obvious that $\omega_T(k)$ is a nonnegative, non-increasing function on $[k_0, +\infty)$. Moreover, the uniform bound (2.19) implies that $\omega_T(k)$ is bounded for any $T > 0$.

For any $k \geq k_0$, multiplying the first and second equation in (2.1) by $(n_\sigma - k)^+$ and $(p_\sigma - k)^+$, respectively, integrating on \mathbb{R}^3 and adding the resultants together, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|(n_\sigma - k)^+\|_{L^2(\mathbb{R}^3)}^2 + \|(p_\sigma - k)^+\|_{L^2(\mathbb{R}^3)}^2 \right) + \|\nabla(n_\sigma - k)^+\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 \\ & + \|\nabla(p_\sigma - k)^+\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 \\ & = \int_{\mathbb{R}^3} [\operatorname{div}(n_\sigma \nabla(\psi_\sigma + V_n))(n_\sigma - k)^+ + \operatorname{div}(p_\sigma \nabla(-\psi_\sigma + V_p))(p_\sigma - k)^+] dx \\ & - \int_{\mathbb{R}^3} R \left(\frac{n_\sigma}{1 + \sigma n_\sigma}, \frac{p_\sigma}{1 + \sigma p_\sigma}, x \right) [(n_\sigma - k)^+ + (p_\sigma - k)^+] dx \\ & =: I_1 + I_2. \end{aligned} \quad (3.8)$$

Integrating by parts and using the Poisson equation for ψ_σ , we expand the first term I_1 as follows

$$I_1 = k \int_{\mathbb{R}^3} [\Delta(\psi_\sigma + V_n)(n_\sigma - k)^+ + \Delta(-\psi_\sigma + V_p)(p_\sigma - k)^+] dx$$

$$\begin{aligned}
& + \int_{\mathbb{R}^3} (\operatorname{div}[(n_\sigma - k)^+ \nabla(\psi_\sigma + V_n)](n_\sigma - k)^+ + \operatorname{div}[(p_\sigma - k)^+ \nabla(-\psi_\sigma + V_p)](p_\sigma - k)^+) dx \\
= & k \int_{\mathbb{R}^3} (-n_\sigma + p_\sigma + D(x)) [(n_\sigma - k)^+ - (p_\sigma - k)^+] dx \\
& + k \int_{\mathbb{R}^3} (\Delta V_n (n_\sigma - k)^+ + \Delta V_p (p_\sigma - k)^+) dx \\
& - \frac{1}{2} \int_{\mathbb{R}^3} \nabla(\psi_\sigma + V_n) \cdot \nabla[(n_\sigma - k)^+]^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} \nabla(-\psi_\sigma + V_p) \cdot \nabla[(p_\sigma - k)^+]^2 dx \\
= & k \int_{\mathbb{R}^3} (-n_\sigma + p_\sigma) [(n_\sigma - k)^+ - (p_\sigma - k)^+] dx \\
& + k \int_{\mathbb{R}^3} D(x) [(n_\sigma - k)^+ - (p_\sigma - k)^+] dx \\
& + k \int_{\mathbb{R}^3} (\Delta V_n (n_\sigma - k)^+ + \Delta V_p (p_\sigma - k)^+) dx \\
& + \frac{1}{2} \int_{\mathbb{R}^3} (-n_\sigma + p_\sigma) ([(n_\sigma - k)^+]^2 - [(p_\sigma - k)^+]^2) dx \\
& + \frac{1}{2} \int_{\mathbb{R}^3} D(x) ([(n_\sigma - k)^+]^2 - [(p_\sigma - k)^+]^2) dx \\
& + \frac{1}{2} \int_{\mathbb{R}^3} (\Delta V_n [(n_\sigma - k)^+]^2 + \Delta V_p [(p_\sigma - k)^+]^2) dx \\
=: & J_1 + \dots + J_6. \tag{3.9}
\end{aligned}$$

It is easy to verify that

$$J_1 \leq 0, \quad J_4 \leq 0, \quad \forall k \geq k_0. \tag{3.10}$$

Besides, it follows from (H2b) that

$$I_2 \leq c_2 \int_{\mathbb{R}^3} \mu_n \mu_p (1 + n_\sigma + p_\sigma) [(n_\sigma - k)^+ + (p_\sigma - k)^+] dx := J_7. \tag{3.11}$$

Integrating (3.8) with respect to time, we infer from (3.9), (3.10) and (3.11) that

$$\begin{aligned}
& \frac{1}{2} \left(\| (n_\sigma(t) - k)^+ \|_{L^2(\mathbb{R}^3)}^2 + \| (p_\sigma(t) - k)^+ \|_{L^2(\mathbb{R}^3)}^2 \right) \\
& + \int_0^t (\| \nabla(n_\sigma - k)^+ \|_{\mathbf{L}^2(\mathbb{R}^3)}^2 + \| \nabla(p_\sigma - k)^+ \|_{\mathbf{L}^2(\mathbb{R}^3)}^2) d\tau \\
\leq & \int_0^t (J_2 + J_3 + J_5 + J_6 + J_7) d\tau. \tag{3.12}
\end{aligned}$$

By assumptions (H1b) and (H3), we have

$$\begin{aligned}
& \left| \int_0^t J_5 d\tau \right| + \left| \int_0^t J_6 d\tau \right| \\
\leq & \frac{1}{2} \|D(x)\|_{L^\infty(\mathbb{R}^3)} \int_0^t \| (n_\sigma - k)^+ \|_{L^2(\mathbb{R}^3)}^2 + \| (p_\sigma - k)^+ \|_{L^2(\mathbb{R}^3)}^2 d\tau \\
& + \|\Delta V_n\|_{L^\infty(\mathbb{R}^3)} \int_0^t \| (n_\sigma - k)^+ \|_{L^2(\mathbb{R}^3)}^2 d\tau + \|\Delta V_p\|_{L^\infty(\mathbb{R}^3)} \int_0^t \| (p_\sigma - k)^+ \|_{L^2(\mathbb{R}^3)}^2 d\tau \\
\leq & C \int_0^t \| (n_\sigma - k)^+ \|_{L^2(\mathbb{R}^3)}^2 + \| (p_\sigma - k)^+ \|_{L^2(\mathbb{R}^3)}^2 d\tau. \tag{3.13}
\end{aligned}$$

Furthermore, using the Hölder inequality and Gagliardo–Nirenberg inequality, we get

$$\begin{aligned}
& \left| \int_0^t J_2 d\tau \right| + \left| \int_0^t J_3 d\tau \right| \\
& \leq k \int_0^t \int_{\mathbb{R}^3} (|D(x)| + |\Delta V_n|)(n_\sigma - k)^+ dx d\tau + k \int_0^t \int_{\mathbb{R}^3} (|D(x)| + |\Delta V_p|)(p_\sigma - k)^+ dx d\tau \\
& \leq k(\|D(x)\|_{L^\infty(\mathbb{R}^3)} + \|\Delta V_n\|_{L^\infty(\mathbb{R}^3)}) \int_0^t \|(n_\sigma - k)^+\|_{L^3(\mathbb{R}^3)} |B_{nk}|^{\frac{2}{3}} d\tau \\
& \quad + k(\|D(x)\|_{L^\infty(\mathbb{R}^3)} + \|\Delta V_p\|_{L^\infty(\mathbb{R}^3)}) \int_0^t \|(p_\sigma - k)^+\|_{L^3(\mathbb{R}^3)} |B_{pk}|^{\frac{2}{3}} d\tau \\
& \leq Ck\omega_T(k)^{\frac{2}{3}} \int_0^t \|\nabla(n_\sigma - k)^+\|_{\mathbf{L}^2(\mathbb{R}^3)}^{\frac{1}{2}} \|(n_\sigma - k)^+\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} d\tau \\
& \quad + Ck\omega_T(k)^{\frac{2}{3}} \int_0^t \|\nabla(p_\sigma - k)^+\|_{\mathbf{L}^2(\mathbb{R}^3)}^{\frac{1}{2}} \|(p_\sigma - k)^+\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} d\tau \\
& \leq \eta T \int_0^t \|\nabla(n_\sigma - k)^+\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 + \|\nabla(p_\sigma - k)^+\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 d\tau \\
& \quad + \eta T \int_0^t \|(n_\sigma - k)^+\|_{L^2(\mathbb{R}^3)}^2 + \|(p_\sigma - k)^+\|_{L^2(\mathbb{R}^3)}^2 d\tau \\
& \quad + \frac{C}{\eta} k^2 \omega_T(k)^{\frac{4}{3}}. \tag{3.14}
\end{aligned}$$

Next,

$$\begin{aligned}
& \left| \int_0^t J_7 d\tau \right| \\
& \leq \int_0^t \int_{\mathbb{R}^3} \mu_n \mu_p [(n_\sigma - k)^+ + (p_\sigma - k)^+ + 2k + 1] [(n_\sigma - k)^+ + (p_\sigma - k)^+] dx d\tau \\
& \leq C \int_0^t \|(n_\sigma - k)^+\|_{L^2(\mathbb{R}^3)}^2 + \|(p_\sigma - k)^+\|_{L^2(\mathbb{R}^3)}^2 d\tau \\
& \quad + C(2k + 1) \int_0^t \int_{\mathbb{R}^3} (n_\sigma - k)^+ dx d\tau + C(2k + 1) \int_0^t \int_{\mathbb{R}^3} (p_\sigma - k)^+ dx d\tau \\
& \leq C \int_0^t (\|(n_\sigma - k)^+\|_{L^2(\mathbb{R}^3)}^2 + \|(p_\sigma - k)^+\|_{L^2(\mathbb{R}^3)}^2) d\tau \\
& \quad + C(2k + 1)\omega_T(k)^{\frac{2}{3}} \int_0^t (\|(n_\sigma - k)^+\|_{L^3(\mathbb{R}^3)} + \|(p_\sigma - k)^+\|_{L^3(\mathbb{R}^3)}) d\tau \\
& \leq \eta T \int_0^t \|\nabla(n_\sigma - k)^+\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 + \|\nabla(p_\sigma - k)^+\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 d\tau \\
& \quad + (C + \eta T) \int_0^t \|(n_\sigma - k)^+\|_{L^2(\mathbb{R}^3)}^2 + \|(p_\sigma - k)^+\|_{L^2(\mathbb{R}^3)}^2 d\tau \\
& \quad + \frac{C}{\eta} (2k + 1)^2 \omega_T(k)^{\frac{4}{3}}. \tag{3.15}
\end{aligned}$$

Taking $\eta = \frac{1}{2T}$ in the above estimates, we obtain that

$$\begin{aligned}
& \| (n_\sigma(t) - k)^+ \|_{L^2(\mathbb{R}^3)}^2 + \| (p_\sigma(t) - k)^+ \|_{L^2(\mathbb{R}^3)}^2 \\
& + \int_0^t (\| \nabla (n_\sigma - k)^+ \|_{\mathbf{L}^2(\mathbb{R}^3)}^2 + \| \nabla (p_\sigma - k)^+ \|_{\mathbf{L}^2(\mathbb{R}^3)}^2) d\tau \\
& \leq C_1 \int_0^t \left(\| (n_\sigma(\tau) - k)^+ \|_{L^2(\mathbb{R}^3)}^2 + \| (p_\sigma(\tau) - k)^+ \|_{L^2(\mathbb{R}^3)}^2 \right) d\tau \\
& + TC_2(k^2 + 1)\omega_T(k)^{\frac{4}{3}}.
\end{aligned} \tag{3.16}$$

It follows from the Gronwall inequality that

$$\| (n_\sigma(t) - k)^+ \|_{L^2(\mathbb{R}^3)}^2 + \| (p_\sigma(t) - k)^+ \|_{L^2(\mathbb{R}^3)}^2 \leq e^{C_1 T} TC_2(k^2 + 1)\omega_T(k)^{\frac{4}{3}}, \quad \forall t \in [0, T]. \tag{3.17}$$

On the other hand, for any $t \in [0, T]$ and $\hat{k} > k \geq k_0$,

$$\begin{aligned}
\| (n_\sigma(t) - k)^+ \|_{L^2(\mathbb{R}^3)}^2 &= \int_{B_{n\hat{k}}(t)} |(n_\sigma(t, \cdot) - k)^+|^2 dx \geq \int_{B_{n\hat{k}}(t)} (n_\sigma(t, \cdot) - k)^2 dx \\
&\geq (\hat{k} - k)^2 |B_{n\hat{k}}(t)|,
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
\| (p_\sigma(t) - k)^+ \|_{L^2(\mathbb{R}^3)}^2 &= \int_{B_{p\hat{k}}(t)} |(p_\sigma(t, \cdot) - k)^+|^2 dx \geq \int_{B_{p\hat{k}}(t)} (p_\sigma(t, \cdot) - k)^2 dx \\
&\geq (\hat{k} - k)^2 |B_{p\hat{k}}(t)|.
\end{aligned} \tag{3.19}$$

We deduce from (3.17)–(3.19) that

$$\omega_T(\hat{k}) \leq e^{C_1 T} TC_2(k^2 + 1)(\hat{k} - k)^{-2} \omega_T(k)^{\frac{4}{3}}, \quad \forall \hat{k} > k \geq k_0. \tag{3.20}$$

Now in Lemma 3.1, we set

$$M(k) = e^{C_1 T} TC_2(k^2 + 1) \geq 0, \quad \gamma = 2, \quad \beta = \frac{1}{3}. \tag{3.21}$$

The function $M(k)$ has the following property

$$\frac{M(k)}{k^2} = \frac{k^2 + 1}{k^2} e^{C_1 T} TC_2 \leq \left(1 + \frac{1}{k_0^2} \right) e^{C_1 T} TC_2 := M_0, \quad \forall k \in [k_0, +\infty). \tag{3.22}$$

Therefore, there exists a constant

$$k^* = 2k_0 \left(1 + 2^{15} M_0^2 \omega_T(k_0)^{\frac{2}{3}} \right) > k_0, \tag{3.23}$$

which is independent of σ such that

$$\omega_T(k^*) = 0. \tag{3.24}$$

Namely,

$$n_\sigma(t, x) \leq k^* \quad \text{and} \quad p_\sigma(t, x) \leq k^*, \quad \text{a.e. in } \mathbb{R}^3 \times [0, T]. \tag{3.25}$$

The proof is complete. \square

Proof of Theorem 1.1. Lemma 2.2 and Lemma 3.2 yield the following uniform estimates that are independent of the parameter $\sigma > 0$:

$$\|n_\sigma\|_{L^\infty(0,T;L^q(\mathbb{R}^3))} + \|p_\sigma\|_{L^\infty(0,T;L^q(\mathbb{R}^3))} \leq C_T, \quad (3.26)$$

$$\|n_\sigma\|_{L^2(0,T;H^1(\mathbb{R}^3))} + \|p_\sigma\|_{L^2(0,T;H^1(\mathbb{R}^3))} \leq C_T, \quad (3.27)$$

$$\|\nabla\psi_\sigma\|_{L^\infty(0,T;\mathbf{L}^{q'}(\mathbb{R}^3))} + \|\Delta\psi_\sigma\|_{L^\infty(0,T;L^q(\mathbb{R}^3))} \leq C_T, \quad q \in (0, +\infty), \quad (3.28)$$

where $q \in [1, +\infty]$, $q' \in (1, +\infty]$. Besides, we infer from (2.42), (2.11) and (H1) that

$$\int_0^T \int_{\mathbb{R}^3} (n_\sigma^2 |\nabla V_n|^2 dx + p_\sigma^2 |\nabla V_p|^2) dx dt \leq e^{-V_b} \int_0^T \int_{\mathbb{R}^3} (u^2 |\nabla V_n|^2 dx + v^2 |\nabla V_p|^2) dx dt \leq C_T. \quad (3.29)$$

Then by the equations for n_σ and p_σ in (2.1) and (3.26)–(3.29), we obtain that

$$\|\partial_t n_\sigma\|_{L^2(0,T;(H^1(\mathbb{R}^3))')} + \|\partial_t p_\sigma\|_{L^2(0,T;(H^1(\mathbb{R}^3))')} \leq C_T, \quad (3.30)$$

From the uniform estimates (3.26)–(3.28), we deduce that there exist

$$n, p \in L^\infty(0, T; L^\infty(\mathbb{R}^3)), \quad (3.31)$$

with

$$\nabla u, \nabla p \in L^2(0, T; \mathbf{L}^2(\mathbb{R}^3)), \quad \partial_t u, \partial_t p \in L^2(0, T; (H^1(\mathbb{R}^3))'), \quad (3.32)$$

and ψ with $\Delta\psi \in L^\infty(0, T; L^\infty(\mathbb{R}^3))$, $\nabla\psi \in L^2(0, T; \mathbf{L}^2(\mathbb{R}^3))$ such that for a sequence $\{\sigma_j\} \searrow 0$ as $j \rightarrow +\infty$ (not relabeled when taking a subsequence),

$$n_{\sigma_j} \rightharpoonup n, \quad p_{\sigma_j} \rightharpoonup p, \quad \text{weakly-star in } L^\infty(0, T; L^\infty(\mathbb{R}^3)), \quad (3.33)$$

$$\nabla n_{\sigma_j} \rightharpoonup \nabla n, \quad \nabla p_{\sigma_j} \rightharpoonup \nabla p, \quad \text{weakly in } L^2(0, T; \mathbf{L}^2(\mathbb{R}^3)), \quad (3.34)$$

$$\partial_t n_{\sigma_j} \rightharpoonup \partial_t n, \quad \partial_t p_{\sigma_j} \rightharpoonup \partial_t p, \quad \text{weakly in } L^2(0, T; (H^1(\mathbb{R}^3))'), \quad (3.35)$$

$$\Delta\psi_{\sigma_j} \rightharpoonup \Delta\psi, \quad \text{weakly-star in } L^\infty(0, T; L^\infty(\mathbb{R}^3)), \quad (3.36)$$

$$\nabla\psi_{\sigma_j} \rightharpoonup \nabla\psi, \quad \text{weakly in } L^2(0, T; \mathbf{L}^2(\mathbb{R}^3)). \quad (3.37)$$

Moreover, on account of the compact embedding theorem we have

$$n_{\sigma_j} \rightarrow n, \quad p_{\sigma_j} \rightarrow p, \quad \text{strongly in } L^2(0, T; L_{loc}^2(\mathbb{R}^3)), \quad \text{thus also a.e. in } (0, T) \times \mathbb{R}^3. \quad (3.38)$$

Then, for any $\varphi \in L^2(0, T; C_c^\infty(\mathbb{R}^3))$, we have

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}^3} (n_{\sigma_j} \nabla\psi_{\sigma_j} - n \nabla\psi) \nabla\varphi dx dt \right| \\ & \leq \left| \int_0^T \int_{\mathbb{R}^3} (n_{\sigma_j} - n) \nabla\psi_{\sigma_j} \nabla\varphi dx dt \right| + \left| \int_0^T \int_{\mathbb{R}^3} (\nabla\psi_{\sigma_j} - \nabla\psi) n \nabla\varphi dx dt \right| \\ & \leq \|\nabla\psi_{\sigma_j}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \|n_{\sigma_j} - n\|_{L^2(0,T;L^2(\text{supp}\varphi))} \|\nabla\varphi\|_{L^2(0,T;L^\infty(\mathbb{R}^3))} \\ & \quad + \left| \int_0^T \int_{\mathbb{R}^3} (\nabla\psi_{\sigma_j} - \nabla\psi) (n \nabla\varphi) dx dt \right| \\ & \rightarrow 0, \quad \text{as } \sigma_j \rightarrow 0, \end{aligned} \quad (3.39)$$

similarly,

$$\left| \int_0^T \int_{\mathbb{R}^3} (n_{\sigma_j} \nabla \psi_{\sigma_j} - n \nabla \psi) \nabla \varphi dx dt \right| \rightarrow 0, \quad \text{as } \sigma_j \rightarrow 0. \quad (3.40)$$

Next, we study the convergence of the recombination-generation rate. The uniform bound (3.26) and (H2b) yield that

$$\left\| \tilde{R}(n_{\sigma}, p_{\sigma}, x) \right\|_{L^2(0, T; L^2(\mathbb{R}^3))} \leq C_T. \quad (3.41)$$

Thus there exists $G \in L^2(0, T; L^2(\mathbb{R}^3))$ such that (up to a subsequence)

$$\tilde{R}(n_{\sigma}, p_{\sigma}, x) \rightarrow G, \quad \text{weakly in } L^2(0, T; L^2(\mathbb{R}^3)) \quad \text{as } \sigma_j \rightarrow 0. \quad (3.42)$$

For any bounded domain $\Omega \subset \mathbb{R}^3$, there holds

$$\begin{aligned} \int_0^T \int_{\Omega} \left(\frac{n_{\sigma_j}}{1 + \sigma_j n_{\sigma_j}} - n \right)^2 dx dt &\leq \int_0^T \int_{\Omega} (n_{\sigma_j} - n)^2 dx dt + \int_0^T \int_{\Omega} (\sigma_j n_{\sigma_j} n)^2 dx dt \\ &\rightarrow 0, \quad \text{as } \sigma_j \rightarrow 0, \end{aligned} \quad (3.43)$$

which implies that

$$\frac{n_{\sigma_j}}{1 + \sigma_j n_{\sigma_j}} \rightarrow n \quad \text{strongly in } L^2(0, T; L^2_{loc}(\mathbb{R}^3)). \quad (3.44)$$

In the same manner, we have

$$\frac{p_{\sigma_j}}{1 + \sigma_j p_{\sigma_j}} \rightarrow p \quad \text{strongly in } L^2(0, T; L^2_{loc}(\mathbb{R}^3)). \quad (3.45)$$

Since F is Lip-continuous (see (H2b)), we infer from (3.44) and (3.45) that on any bounded domain $\Omega \subset \mathbb{R}^3$,

$$\begin{aligned} &\int_0^T \int_{\Omega} \left[F \left(\frac{n_{\sigma_j}}{1 + \sigma_j n_{\sigma_j}}, \frac{p_{\sigma_j}}{1 + \sigma_j p_{\sigma_j}} \right) - F(n, p) \right]^2 dx dt \\ &\leq C \int_0^T \int_{\Omega} \left(\frac{n_{\sigma_j}}{1 + \sigma_j n_{\sigma_j}} - n \right)^2 + \left(\frac{p_{\sigma_j}}{1 + \sigma_j p_{\sigma_j}} - p \right)^2 dx dt \\ &\rightarrow 0, \quad \text{as } \sigma_j \rightarrow 0, \end{aligned} \quad (3.46)$$

namely,

$$F \left(\frac{n_{\sigma_j}}{1 + \sigma_j n_{\sigma_j}}, \frac{p_{\sigma_j}}{1 + \sigma_j p_{\sigma_j}} \right) \rightarrow F(n, p) \quad \text{strongly in } L^2(0, T; L^2_{loc}(\mathbb{R}^3)) \quad \text{and a.e. in } (0, T) \times \mathbb{R}^3. \quad (3.47)$$

As a result, we have the point-wise convergence of \tilde{R}

$$\tilde{R}(n_{\sigma}, p_{\sigma}, x) \rightarrow R(n, p, x), \quad \text{a.e. in } (0, T) \times \mathbb{R}^3, \quad (3.48)$$

which together with (3.42) implies that $G = R(n, p, x)$ and

$$\tilde{R}(n_{\sigma}, p_{\sigma}, x) \rightharpoonup R(n, p, x), \quad \text{weakly in } L^2(0, T; L^2(\mathbb{R}^3)) \quad \text{as } \sigma_j \rightarrow 0. \quad (3.49)$$

Based on the above convergent results, now we are able to pass to the limit by letting $\sigma_j \rightarrow 0$ in the approximate problem (2.1) and obtain a global solution (n, p, ψ) of problem (1.1)–(1.2). The system (1.1) is satisfied in the following sense that for any $\varphi \in L^2(0, T; H^1(\mathbb{R}^3))$,

$$\int_0^T \langle n_t, \varphi \rangle_{H^{-1}, H^1} dt + \int_0^T \int_{\mathbb{R}^3} (\nabla n + n \nabla(\psi + V_n)) \nabla \varphi dx dt + \int_0^T \int_{\mathbb{R}^3} R(n, p, x) \varphi dx dt = 0, \quad (3.50)$$

$$\int_0^T \langle p_t, \varphi \rangle_{H^{-1}, H^1} dt + \int_0^T \int_{\mathbb{R}^3} (\nabla p + p \nabla(-\psi + V_p)) \nabla \varphi dx dt + \int_0^T \int_{\mathbb{R}^3} R(n, p, x) \varphi dx dt = 0, \quad (3.51)$$

$$\int_0^T \int_{\mathbb{R}^3} \nabla \psi \cdot \nabla \varphi dx dt = \int_0^T \int_{\mathbb{R}^3} (n - p - D) \varphi dx dt. \quad (3.52)$$

Finally, we prove the uniqueness of global solutions to problem (1.1)–(1.2). Let (n_i, p_i, ψ_i) ($i = 1, 2$) be two solutions to problem (1.1)–(1.2) with initial data n_{Ii}, p_{Ii} . Set $n = n_1 - n_2$, $p = p_1 - p_2$, $\psi = \psi_1 - \psi_2$, $n_I = n_{I1} - n_{I2}$ and $p_I = p_{I1} - p_{I2}$. Taking the difference of the equations for n_1 and n_2 , testing the resultant by n , we find that

$$\begin{aligned} & \frac{1}{2} \|n(t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \|\nabla n\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 d\tau \\ &= \frac{1}{2} \|n_I\|_{L^2(\mathbb{R}^3)}^2 - \int_0^t \int_{\mathbb{R}^3} n \nabla n \cdot \nabla V_n dx d\tau - \int_0^t \int_{\mathbb{R}^3} (n \nabla \psi_1 + n_2 \nabla \psi) \nabla n dx d\tau \\ & \quad - \int_0^t \int_{\mathbb{R}^3} (R(n_1, p_1, x) - R(n_2, p_2, x)) n dx d\tau \\ &:= \frac{1}{2} \|n_I\|_{L^2(\mathbb{R}^3)}^2 + E_1 + E_2 + E_3. \end{aligned} \quad (3.53)$$

Using the uniform estimates for n_i, p_i, ψ_i , we have

$$\begin{aligned} |E_1 + E_2| &\leq \frac{1}{2} \left(\|\Delta V_n\|_{L^\infty(\mathbb{R}^3)} + \|\Delta \psi_1\|_{L^\infty(\mathbb{R}^3)} \right) \int_0^t \|n\|_{L^2(\mathbb{R}^3)}^2 d\tau \\ &\quad + \|n_2\|_{L^\infty(0, T; L^3(\mathbb{R}^3))} \int_0^t \left(\epsilon \|\nabla n\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 + C_\epsilon \|\nabla \psi\|_{\mathbf{L}^6(\mathbb{R}^3)}^2 \right) d\tau. \end{aligned} \quad (3.54)$$

Then by (H2a)–(H2b), we deduce that

$$\begin{aligned} |E_3| &\leq \left| \int_0^t \int_{\mathbb{R}^3} (F(n_1, p_1) - F(n_2, p_2)) n \mu_n \mu_p dx d\tau \right| \\ &\quad + \left| \int_0^t \int_{\mathbb{R}^3} (F(n_1, p_1) - F(n_2, p_2)) n_1 p_1 n dx d\tau \right| \\ &\quad + \left| \int_0^t \int_{\mathbb{R}^3} F(n_2, p_2) (n_1 p + n p_2) n dx d\tau \right| \\ &\leq C \left(\|V_b\|, \|n_i\|_{L^\infty(0, T; L^\infty(\mathbb{R}^3))}, \|p_i\|_{L^\infty(0, T; L^\infty(\mathbb{R}^3))} \right) \\ &\quad \times \int_0^t (\|n\|_{L^2(\mathbb{R}^3)}^2 + \|p\|_{L^2(\mathbb{R}^3)}^2) d\tau. \end{aligned} \quad (3.55)$$

In a similar manner, we have the following estimate for p

$$\frac{1}{2} \|p\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \|\nabla p\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 d\tau$$

$$\begin{aligned}
&\leq \frac{1}{2} \|p_I\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \left(\|\Delta V_p\|_{L^\infty(\mathbb{R}^3)} + \|\Delta \psi_1\|_{L^\infty(\mathbb{R}^3)} \right) \int_0^t \|p\|_{L^2(\mathbb{R}^3)}^2 d\tau \\
&\quad + \|p_2\|_{L^\infty(0,T;L^3(\mathbb{R}^3))} \int_0^t \left(\epsilon \|\nabla p\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 + C_\epsilon \|\nabla \psi\|_{\mathbf{L}^6(\mathbb{R}^3)}^2 \right) d\tau \\
&\quad + C(V_b, \|n_i\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}, \|p_i\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}) \int_0^t (\|n\|_{L^2(\mathbb{R}^3)}^2 + \|p\|_{L^2(\mathbb{R}^3)}^2) d\tau. \tag{3.56}
\end{aligned}$$

Since ψ satisfies the Poisson equation $-\Delta \psi = n - p$, then it follows from [19, Corollary 2.2] that

$$\|\nabla \psi\|_{\mathbf{L}^6(\mathbb{R}^3)}^2 \leq C \left(\|n\|_{L^2(\mathbb{R}^3)}^2 + \|p\|_{L^2(\mathbb{R}^3)}^2 \right). \tag{3.57}$$

Therefore, taking ϵ sufficiently small satisfying

$$0 < \epsilon \leq \frac{1}{2} \min \left\{ 1, \|n_2\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}^{-1}, \|p_2\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}^{-1} \right\}, \tag{3.58}$$

we deduce from (3.53)–(3.57) that

$$\begin{aligned}
&\|n(t)\|_{L^2(\mathbb{R}^3)}^2 + \|p(t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \left(\|\nabla n\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 + \|\nabla p\|_{\mathbf{L}^2(\mathbb{R}^3)}^2 \right) d\tau \\
&\leq \|n_I\|_{L^2(\mathbb{R}^3)}^2 + \|p_I\|_{L^2(\mathbb{R}^3)}^2 + C_T \int_0^t \left(\|n\|_{L^2(\mathbb{R}^3)}^2 + \|p\|_{L^2(\mathbb{R}^3)}^2 \right) d\tau. \tag{3.59}
\end{aligned}$$

From the Gronwall inequality, we can conclude the continuous dependence on the initial data that

$$\|n(t)\|_{L^2(\mathbb{R}^3)}^2 + \|p(t)\|_{L^2(\mathbb{R}^3)}^2 \leq \left(\|n_I\|_{L^2(\mathbb{R}^3)}^2 + \|p_I\|_{L^2(\mathbb{R}^3)}^2 \right) e^{C_T t}, \quad \forall t \in [0, T], \tag{3.60}$$

which yields the uniqueness. The proof is complete.

4 Remark on the Long-time Behavior

In the last section, we have proved the existence and uniqueness of global solutions to problem (1.1)–(1.2). However, the global-in-time estimates for the solution (n, p, ψ) depends on T that can be chosen arbitrary. In this section, we extend the results in [10, 24] to our current problem (1.1)–(1.2). For this purpose, we first need to obtain some uniform-in-time estimates on the global solution.

Let $\alpha = \int_{\mathbb{R}^3} (n_I - p_I) dx$. We easily see from (1.1) that the difference of mass is conserved for all $t > 0$:

$$\int_{\mathbb{R}^3} (n(t, \cdot) - p(t, \cdot)) dx = \alpha. \tag{4.1}$$

The relative entropy associated with (1.1) is as follows:

$$\begin{aligned}
e(t) &:= \int_{\mathbb{R}^3} \left[n \left(\ln \frac{n}{n_\infty} - 1 \right) + n_\infty \right] dx + \int_{\mathbb{R}^3} \left[p \left(\ln \frac{p}{p_\infty} - 1 \right) + p_\infty \right] dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \psi - \nabla \psi_\infty|^2 dx, \tag{4.2}
\end{aligned}$$

where $(n_\infty, p_\infty, \psi_\infty)$ is the steady state of system (1.1) that satisfies

$$\begin{cases} n_\infty(x) = D_n e^{-\psi_\infty} \mu_n, & D_n \in \mathbb{R}^+, \\ p_\infty(x) = D_p e^{\psi_\infty} \mu_p, & D_p \in \mathbb{R}^+, \\ n_\infty p_\infty = \mu_n \mu_p, & \int_{\mathbb{R}^3} n_\infty dx - \int_{\mathbb{R}^3} p_\infty dx = \alpha, \\ -\Delta \psi_\infty = n_\infty - p_\infty - D(x). \end{cases} \quad (4.3)$$

Remark 4.1. Denote $I = \int_{\mathbb{R}^3} e^{-\psi_\infty - V_n(x)} dx$, $J = \int_{\mathbb{R}^3} e^{\psi_\infty - V_p(x)} dx$. Then the coefficients D_n and D_p in (4.3) are given by (cf. [24, Lemma 3.1])

$$D_n = \frac{\alpha + \sqrt{\alpha^2 + 4IJ}}{2I}, \quad D_p = \frac{\sqrt{\alpha^2 + 4IJ} - \alpha}{2J}. \quad (4.4)$$

Following the argument in [24, Theorem 3.1], where the special case $V_n = V_p = V$ was considered, we can still prove the existence and uniqueness of $(n_\infty, p_\infty, \psi_\infty)$.

Proposition 4.1. Suppose that assumptions (H1a), (H1b) and (H3) are satisfied. Then the stationary problem (4.3) admits a unique solution $(\psi_\infty, n_\infty, p_\infty)$ such that $\psi_\infty \in D^{1,2}(\mathbb{R}^3) = \{\phi(x) \in L^6(\mathbb{R}^3) \mid \nabla \phi \in \mathbf{L}^2(\mathbb{R}^3)\}$. Moreover, $\psi_\infty \in L^\infty(\mathbb{R}^3)$ and $\nabla \psi_\infty \in \mathbf{L}^\infty(\mathbb{R}^3) \cap \mathbf{L}^2(\mathbb{R}^3)$.

The we have

Lemma 4.1. Suppose the assumptions of Theorem 1.1 are satisfied. The global solution (n, p) of problem (1.1)–(1.2) satisfies

$$\sup_{t \geq 0} [\|n(t)\|_{L^r(\mathbb{R}^3)} + \|p(t)\|_{L^r(\mathbb{R}^3)}] < \infty, \quad \forall r \in [1, +\infty]. \quad (4.5)$$

Proof. By a straightforward calculation, we have the dissipation of the relative entropy

$$\frac{d}{dt} e(t) = -\mathcal{D}(t) \leq 0, \quad (4.6)$$

with the entropy dissipation

$$\mathcal{D}(t) = - \int_{\mathbb{R}^3} n \left| \nabla \ln \left(\frac{n}{N} \right) \right|^2 dx - \int_{\mathbb{R}^3} p \left| \nabla \ln \left(\frac{p}{P} \right) \right|^2 dx - \int_{\mathbb{R}^3} R(n, p, x) \ln \left(\frac{np}{\mu_n \mu_p} \right) dx, \quad (4.7)$$

$$\text{where } N = D_n e^{-\psi(t)} \mu_n, \quad P = D_p e^{\psi(t)} \mu_p. \quad (4.8)$$

Based on the entropy dissipation inequality, we can obtain uniform L^r bounds ($r \in [1, +\infty)$) for $n(t)$ and $p(t)$ exactly as in [24, Lemma 4.1]. It only remains to show the uniform L^∞ estimate. We note that L^∞ bounds of solutions to a simplified drift-diffusion system (without self-consistent potential ψ and with a recombination-generation rate of Shockley–Read–Hall type) have been obtained in [10] via a Nash–Moser type iteration method and the results could be extended to the case with self-consistent potential [11]. For the convenience of the readers, we sketch the proof for our present case with a more general recombination-generation rate.

For $r \geq 2$, using integration by parts and the nonnegativity of n, p , we get

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^3} (n^{r+1} + p^{r+1}) dx + \frac{4r}{r+1} \int_{\mathbb{R}^3} \left(\left| \nabla \left(n^{\frac{r+1}{2}} \right) \right|^2 + \left| \nabla \left(p^{\frac{r+1}{2}} \right) \right|^2 \right) dx \\
&= r \int_{\mathbb{R}^3} \Delta \psi (n^{r+1} - p^{r+1}) + r \int_{\mathbb{R}^3} (\Delta V_n n^{r+1} + \Delta V_p p^{r+1}) dx \\
&\quad - (r+1) \int_{\mathbb{R}^3} R(n, p, x) (n^r + p^r) dx \\
&= -r \int_{\mathbb{R}^3} (n - p) (n^{r+1} - p^{r+1}) dx + r \int_{\mathbb{R}^3} D(x) (n^{r+1} - p^{r+1}) dx \\
&\quad + r \int_{\mathbb{R}^3} (\Delta V_n n^{r+1} + \Delta V_p p^{r+1}) dx \\
&\quad - (r+1) \int_{\mathbb{R}^3} F(n, p) (n^{r+1} p + n p^{r+1}) dx + (r+1) \int_{\mathbb{R}^3} \mu^2 F(n, p) (n^r + p^r) dx \\
&\leq r (\|D\|_{L^\infty(\mathbb{R}^3)} + \|\Delta V_n\|_{L^\infty(\mathbb{R}^3)}) \int_{\mathbb{R}^3} n^{r+1} dx + r (\|D\|_{L^\infty(\mathbb{R}^3)} + \|\Delta V_p\|_{L^\infty(\mathbb{R}^3)}) \int_{\mathbb{R}^3} p^{r+1} dx \\
&\quad + C(r+1) \int_{\mathbb{R}^3} (1 + n + p) (n^r + p^r) dx \\
&\leq C(r+1) \int_{\mathbb{R}^3} (n^{r+1} + p^{r+1}) dx + C(r+1) \int_{\mathbb{R}^3} (n^r + p^r) dx, \tag{4.9}
\end{aligned}$$

where we use the facts that

$$(n - p)(n^{r+1} - p^{r+1}) \geq 0, \quad F(n, p)(n^{r+1} p + n p^{r+1}) \geq 0. \tag{4.10}$$

Since

$$(r-1)r^{\frac{2-r}{r-1}} > 0 \text{ for } r \geq 2, \quad \lim_{r \rightarrow +\infty} (r-1)r^{\frac{2-r}{r-1}} = 1, \tag{4.11}$$

by the Young's inequality and the uniform L^1 estimate of n, p , we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^3} (n^r + p^r) dx &\leq \frac{1}{r^2} \int_{\mathbb{R}^3} (n + p) dx + (r-1)r^{\frac{2-r}{r-1}} \int_{\mathbb{R}^3} (n^{r+1} + p^{r+1}) dx \\
&\leq \frac{C}{r^2} + C' \int_{\mathbb{R}^3} (n^{r+1} + p^{r+1}) dx, \quad \forall r \geq 2, \tag{4.12}
\end{aligned}$$

where the constants C, C' are independent of r . Therefore, it follows from (4.9) and (4.12) that

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}^3} (n^{r+1} + p^{r+1}) dx + \frac{4r}{r+1} \int_{\mathbb{R}^3} \left(\left| \nabla \left(n^{\frac{r+1}{2}} \right) \right|^2 + \left| \nabla \left(p^{\frac{r+1}{2}} \right) \right|^2 \right) dx \\
&\leq C(r+1) \int_{\mathbb{R}^3} (n^{r+1} + p^{r+1}) dx + \frac{C}{r}, \tag{4.13}
\end{aligned}$$

where C is independent of r . Based on the differential inequality (4.13), we can argue as in [10, Supplement Lemma 5.1] to obtain the uniform L^∞ bounds for $n(t)$ and $p(t)$. The proof is complete. \square

Proof of Theorem 1.2. Lemma 4.1 yields the uniform-in-time L^r estimates (1.9). Then the conclusion of Theorem 1.2 follows from the same argument as in [24, Theorem 4.1]. The proof is complete.

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